Department of Mathematics
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Discrete Mathematics

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CHAPTER FIVE:

RELATIONS

SECTION 5.5:
Order Relations.

Definition:
A relation R on a set A is called **partial relation** if it reflexive, transitive, and anti-symmetric.
The set A under this relation is called partially ordered set (poset)

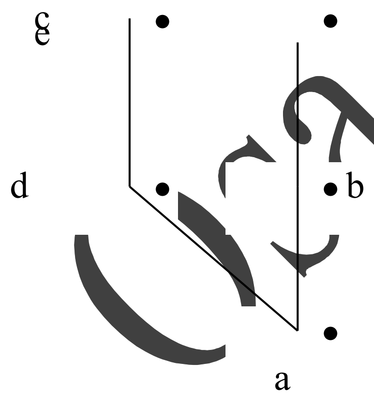
Example:
Let A = Z and define the relation R on A by ${}_xR_y$ if $x \le y$

1. $x \le x$, for all $x \in z$
   $\Rightarrow {}_xR_x$ for all $x \in z$
   $\therefore$ reflexive
2. $x \le y$ and $y \le z \Rightarrow x \le z$
   $\therefore$ transitive
3. $x \le y$ and $y \le x \Rightarrow x = y$
   $\therefore$ anti-symmetric

$\therefore$ R is partial order and z is a poset
If ${}_xR_y$ then we write $x \le y$ and say **x precedes y**.

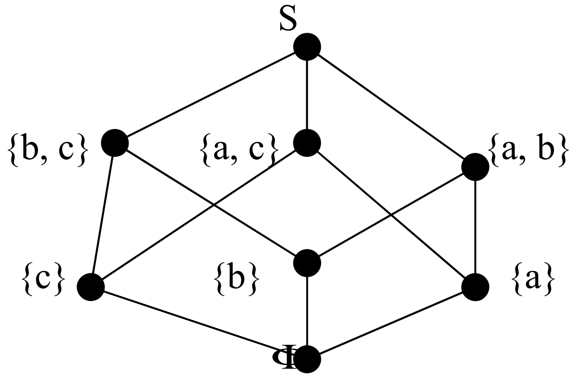
Example:
Let A = { a, b, c, d, e } and
R = { (a, a), (b, b), (c, c), (d, d), (e, e) (a, b), (b, c), (a, c) (a, d), (d, e), (a, e) }
R is partial order on A

- the Hasse diagram of this relation is given by:

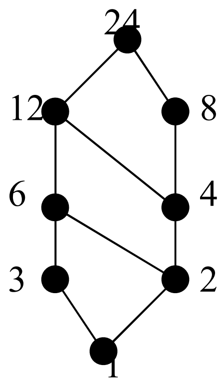


An order relation can be represented by a diagram called Hasse diagram we omit the self loops and the arrows implied be transitivity and if $(a, b) \in R$, then b is above a.

Example:
Let S = { a, b, c }, then [P (S) ∈ , ⊆] is a poset $_{A}R_{B}$ iff A ⊆ B
P (S) = { Φ, S, {a}, {b}, {c}, {a, b}, {a, c}, {b, c} }



Example:
Let A = { 1, 2, 3, 4, 6, 8, 12, 24 } and let ≤ be the relation defined by x ≤ y (x precedes y) iff x|y (x divise y), draw the Hasse diagram for this relation.

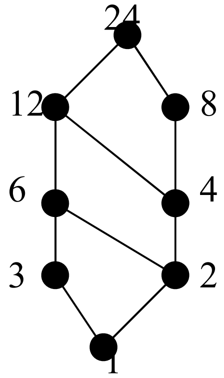


Definition:
A partial order ≤ (precedes) on a set A is said to be total order if every pair of elements x, y ∈ A either x ≤ y or y ≤ x (x precedes y), in this case the set A is called totally ordered set under this relation.

Definition:
Suppose A is a poset under ≤(precedes) and suppose that t ∈ A and x ≤ t (x precedes t) for all x ∈ A we call t greatest elements and suppose that w ∈ A and w ≤ x (w precedes x) for all x ∈ A we call w least elements.

Definition:
A totally ordered set is called well – ordered iff every subset of A has a least element.

Example:
Let A = { 1, 2, 3, 4, 6, 8, 12, 24 } and let $\leq$ be the relation defined by x $\leq$ y (x precedes y) iff x|y (x divise y), draw the Hasse diagram for this relation.



B = { 1, 2, 3, 4, 6, 8, 12, 24 } → called totally ordered

C = { 1, 2, 6,12 } → called NOT totally ordered

D = { 1, 2, 4, 6, 8} → called NOT totally ordered
1: least element
has NO greatest element

E = { 3, 6, 8, 12, 24 } → called NOT totally ordered
Has NO least element
24: greatest element

Example:
(N , $\leq$ (precedes)) totally order, and well ordered
1: least element
has NO greatest element

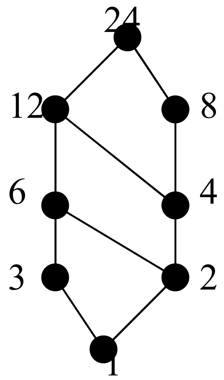
Example:
(Z , $\leq$ (precedes)) totally order, and NOT well ordered
Has NO least element
And { x $\in$ Z: x $\leq$ 0 }

Definition:
Let A be a poset under $\leq$(precedes), let S be a subset of A, an element x $\in$ A is called on upper bound of S iff S $\leq$ x(S precedes x) for all s $\in$ S.
An element y $\in$ A is called a lower bound of S iff y $\leq$ S (y precedes S) for all s $\in$ S.

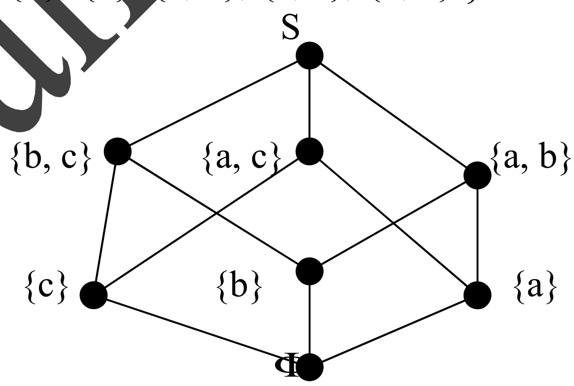
(A, $\leq$) poset and S $\subseteq$ A

Definition:
An element z ∈ A is called **least upper bound** of S if z is an upper bound of S, and z $\leq$ x (z precedes x) for all upper bound x, denoted by **L.u.b** S
An element w ∈ A is called **greatest lower bound** of S if w is a lower bound of S, and x $\leq$ w (x precedes w) for all lower bound x, denoted by **g.l.b** S

Example:
Let A = { 1, 2, 3, 4, 6, 8, 12, 24 } and let $\leq$ be the relation defined by x $\leq$ y (x precedes y)   iff x|y (x divise y).
S = { 2, 4, 12 }
The upper bound of S = { 12, 24 }
The least upper bound (L.u.b) = { 12 }
The lower bound of S = { 1, 2 }
The greatest lower bound (g.l.b) = { 2 }



Example:
Let S = { a, b, c }, then [P (S) ∈ , ⊆] is a poset $_{A}R_{B}$ iff A ⊆ B
P (S) = { **Φ**, S, {a}, {b}, {c}, {a, b}, {a, c}, {b, c} }



S = { {b}, {b, c} }
The upper bound of S = { A, {b, c} }
The least upper bound (L.u.b) = { a, b }
The lower bound of S = { **Φ**, {b} }
The greatest lower bound (g.l.b) = { {b}}

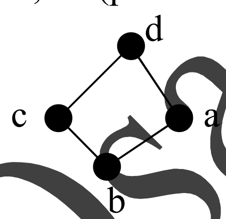
Example:
Let (N , ≤ (precedes)) be a poset
Let S = { 4, 5 }
The upper bound of S = { 5, 6, 7, 8, ……….. }
The least upper bound (L.u.b) = { 5 }
The lower bound of S = { 1, 2, 3, 4 }
The greatest lower bound (g.l.b) = { 4 }

Example:
Let (R , ≤ (precedes)) be a poset
Let S = { x: 0 < x < 1 }
The upper bound of S = { x: 1 ≤ x } = [1, ∞)
The least upper bound (L.u.b) = { 1 }
The lower bound of S = (- ∞, 0]
The greatest lower bound (g.l.b) = { 0 }

Definition:
A poset (A, ≤ (precedes)) is called Lattice iff every subset of exactly two elements has a greatest lower bound (g.l.b) and least upper bound (L.u.b)

Example:
(R , ≤ (precedes))
(N , ≤ (precedes)) are all Lattices
(Z , ≤ (precedes))

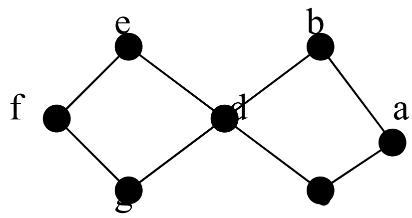
Example:
Let (A , ≤ (precedes)) be a poset with the diagram



is Lattice

The least upper bound (L.u.b) for (a, b) = { a }
The greatest lower bound (g.l.b) for (a, b) = { d }

Example:
Let the diagram of a poset be given by:
Not Lattice
Has no least upper bound (L.u.b)
Has no greatest lower bound (g.l.b)



Example:
Let (N , ≤ (precedes)) and let n ≤ m (n precedes m) iff n|m (n divise m).
S = { 2, 3 }
The upper bound of S = { 6, 12, 18, 24, ……………… }
The least upper bound (L.u.b) = { 6 }
The lower bound of S = { 1 }
The greatest lower bound (g.l.b) = { 1 }

Clarification of the previous example:
Let S = { a, b }
The upper bound of S = { ab, 2ab, 3ab, 4ab, …………………. }
The least upper bound (L.u.b) = { a, b }
The greatest lower bound (g.l.b) = { a, b }

The least upper bound (L.u.b) = Least common multiple (L.C.M) = { a, b }
The greatest lower bound (g.l.b) = Greatest common divisor (G.C.D) = { a, b }

Definition:
Let (A , ≤ (precedes)) Lattice and x, y ∈ A then:
We denoted least upper bound (L.u.b) of {x, y} by $x \vee y$ and call is join of x and y, and we denoted greatest lower bound (g.l.b) of {x, y} by $x \wedge y$ and call it the meet of x and y

Theorem:
Let A be a lattice, and x, y, z ∈ A, then

a. $x \wedge x$
b. $x = x$
c. $x \vee y = y \vee x$
d. $x \wedge y = y \wedge x$
e. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
f. $x \vee (y \vee z) = (x \vee y) \vee z$

Example:
Let (P (A), ≤ (precedes)) is a poset
S, T ∈ A
$S \wedge T$ = greatest lower bound (g.l.b) = (S ∩ T)
$S \vee T$ = least upper bound (L.u.b) = (S ∪ T)