Department of Mathematics
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Discrete Mathematics

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CHAPTER FIVE:

RELATIONS

SECTION 5.4:
Equivalence Classes.

Definitions:

Let A be a set, and let R be an equivalence relation on A, then for each $x \in A$.
$E(x) = \{y: y \in A, {}_yR_x\}$ is called the **equivalence class of x**.

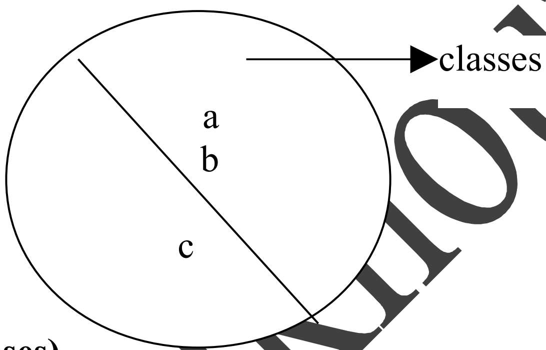
Example:

Let $A = \{a, b, c\}$ $R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$

$E(a) = \{y \in A: {}_yR_a\}$
$= \{y \in A: (y, a) \in R\}$
$= \{a, b\}$

$E(b) = \{y \in A: {}_yR_b\}$
$= \{y \in A: (y, b) \in R\}$
$= \{a, b\}$

$E(c) = \{y \in A: {}_yR_c\}$
$= \{y \in A: (y, c) \in R\}$
$= \{c\}$



➔ **E (a) = E (b) (the same classes)**

Example:

Let $A = \{a, b, c, d, e, f\}$
$R = \{(a, a), (a, b), (b, a), (a, c), (c, a), (c, b), (b, c), (c, c), (d, e), (d, d), (e, d), (e, e), (f, f)\}$

$E(a) = \{y \in A: {}_yR_a\}$
$= \{y \in A: (y, a) \in R\}$
$= \{a, b, c\}$

$E(b) = \{y \in A: {}_yR_b\}$
$= \{y \in A: (y, b) \in R\}$ ➔ $= E(a)$
$= \{a, b, c\}$

$E(c) = \{y \in A: {}_yR_c\}$
$= \{y \in A: (y, c) \in R\}$ ➔ $= E(a) = E(b)$
$= \{a, b, c\}$

$E(d) = \{y \in A: {}_yR_d\}$
$= \{y \in A: (y, d) \in R\}$
$= \{e, d\}$

$E(e) = \{y \in A: {}_yR_e\}$
$= \{y \in A: (y, e) \in R\}$ ➔ $= E(d)$
$= \{e, d\}$

$E(f) = \{y \in A: {}_yR_f\}$
$= \{y \in A: (y, f) \in R\}$
$= \{f\}$

Example:

Let A = Z, R_5 the mod 5 relation

$E(0) = \{ y \in Z: {}_yR_0 \}$
$= \{ y \in Z: y = 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

$E(1) = \{ y \in Z: {}_yR_1 \}$
$= \{ y \in Z: y = 1 + 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -14, -9, -4, 1, 6, 11, 16, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

$E(2) = \{ y \in Z: {}_yR_2 \}$
$= \{ y \in Z: y = 2 + 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -13, -8, -3, 2, 7, 12, 17, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

$E(3) = \{ y \in Z: {}_yR_3 \}$
$= \{ y \in Z: y = 3 + 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -12, -7, -2, 3, 8, 13, 18, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

$E(4) = \{ y \in Z: {}_yR_4 \}$
$= \{ y \in Z: y = 4 + 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -11, -6, -1, 4, 9, 14, 19, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

$E(5) = \{ y \in Z: {}_yR_5 \}$
$= \{ y \in Z: y = 5r, r \in R \}$
$= \{ \ldots\ldots\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \}$

➔ E (0) = E (5) (the same classes)

➔ X – Y = 0 (MOD 5)

then defined class by (0 ➔ MOD – 1) (0 - 4)

Proposition:

Suppose that R is a relation on a set A, and let x, y be two set elements of A, then E (x) and E (y) either equal or disjoint.

Proof:

Suppose that $E(x) \cap E(y) \neq \not\subset$

Then $\exists b \in E(x) \cap E(y)$

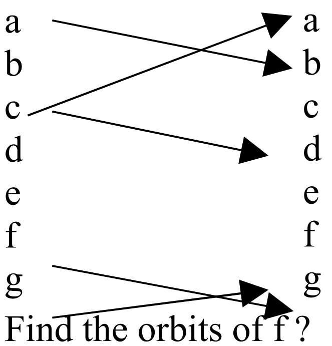
$\Rightarrow b \in E(x)$ and $b \in E(y)$

$\Rightarrow E(b) = E(x) \wedge E(b) = E(y)$

$\Rightarrow E(x) = E(y)$

Definition:
Let A be a finite set and let f be a permutation on A, the equivalence class of R_f are called **Orbits** of the permutation f.

Example:
Let A = { a, b, c, d, e, f, g } and let f: A $\xrightarrow{\text{bijection}}$ A defined by



Find the orbits of f ?

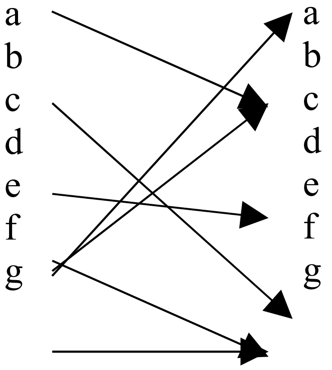
E (a) = { x ∈ A : x R_f a }
= { x ∈ A : f (a) = x for some n ∈ z }
= { b, c, a }
= E (b) = E (c)

E (d) = { x ∈ A : x R_f d }
= { x ∈ A : f (d) = x for some n ∈ z }
= { e, d}
= E (e)

E (f) = { x ∈ A : x R_f f }
= { x ∈ A : f (f) = x for some n ∈ z }
= { g, f}
= E (g)

The Orbits = { b, c, a }, { e, d}, { g, f}
f = (a, b, c), (e, d), (f, g)
f = { b, c, a }, { e, d}, (f, g)
The orbits enables us to write f in a cycle decomposition

Example:
Let A = { a, b, c, d, e, f, b, g } and let g: A $\xrightarrow{\text{bijection}}$ A with cycle decomposition (a, c, d, f, g), (e)



Definition:
Let R be a relation on a set A, a subset of A containing exactly one elements of each equivalence class is called **a complete set of representative** of R

Example:
Let A = Z, find R_5

1. = { 0, 1, 2, 3, 4}
   representative equivalence class
2. = { 5, 1, -3, 8, 19 }
   Complete set of representative