Department of Mathematics
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Discrete Mathematics

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CHAPTER FIVE:

RELATIONS

SECTION 5.3:
Equivalence of Relations.

Definition:
A relation R on a set A is called equivalence relation iff R is reflexive, symmetric, and transitive.

Example:
Let A = {a, b, c }   R = { (a, a), (b, b), (c, c) }
Is an equivalence relation A

Example:
Let A = {a, b, c, d }   R = { (a, a), (b, b), (c, c), (d, d) }
find an equivalence relation on A?
solution:

1. if (a, c) and (b, d) must belongs to R
   R = { (a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (b, d), (d, b) }
2. if (a, c) and (c, d) ∈ R
   R = { (a, a), (b, b), (c, c), (d, d), (a, c), (c, d), (c, a), (d, c), (d, a), (a, d) }

Example:
Let A be the set of integers and define a relation R on Z by $n\, R_5\, m$ if n – m is divisible by 5.
Solution:

Clarification:
$1\, R_5\, 6$ → 1 – 6 = -5, -5 is divisible by 5.
$1\, R_5\, 11$ → 1 – 11 = -10, -10 is divisible by 5.
$1\, R_5\, -4$ → 1 – - 4 = 5, 5 is divisible by 5.
$2\, R_5\, 7$ → 2 – 7 = -5, -5 is divisible by 5.
$2\, R_5\, 12$ → 2 – 12 = -10, -10 is divisible by 5.
$2\, R_5\, -3$ → 2 – - 3 = 5, 5 is divisible by 5.
$2\, R_5\, -13$ → 2 – -13 = -15, -15 is divisible by 5.

Example:
Let A be the set of integers and define a relation R on Z by n R_5 m if n – m is divisible by 5.
Solution:
Is R_5 an equivalence relation.

1. reflexive?

n R_5 n = for all n ∈ Z iff n – n is divisible by 5 → n – n = 0
if 0 is divisible by 5
∴ n R_5 n for all n ∈ Z
∴ R_5 is reflexive

2. symmetric?

$_xR_y$ ∈ **A,** $_yR_x$ ∈ **A**
let n R_5 m ∈ Z (want m R_5 n ∈ Z)
n R_5 m if n – m = 5r for r ∈ Z
m R_5 n if (n – m) * -1 = 5r for r ∈ Z
m R_5 n if m – n = -5r for r ∈ Z
m R_5 n if m – n = 5 (-r) = 5 (s) for s ∈ Z
∴ m R_5 n
∴ R_5 is symmetric

3. transitive?

let n R_5 m ∈ Z and m R_5 t (want n R_5 t ∈ Z)
n – m = 5r
m – t = 5s
n – m + m – t = 5r + 5s
n – t = 5 (r + s)
→ n R_5 t
∴ R_5 is transitive

In general:
For any positive integer m, define a relation R on the set of all integers by $x \underset{m}{R} y$ iff x – y is divisible by m, this relation is an equivalence relation for any integer.

If we have $x \underset{m}{R} y$, then we write $x \equiv y \pmod{m}$ and we say that x is congruent to y modulate m and m is called the modulate of this relation.

Example:

1. Is $9 \underset{5}{R} 4$?

$9 \underset{5}{R} 4$ iff 9 – 4 is divisible by 5

in this case we write $9 \equiv 4 \pmod{5}$

2. Is $23 \underset{5}{R} 13$?

$23 \underset{5}{R} 13$ iff 23 – 13 is divisible by 5

in this case we write $23 \equiv 13 \pmod{5}$

3. is $4 \equiv 10 \pmod{3}$?

yes, $4 \underset{3}{R} 10$

4. is $23 \equiv 42 \pmod{19}$?

$23 \equiv 42 \pmod{19}$, iff $23 \underset{19}{R} 42$ iff 23 – 42 is divisible by 19.

Example:
Solve the following Congruencies?

1. $x + 1 \equiv 3 \pmod{5}$
2. $2x \equiv 4 \pmod{5}$
3. $2x \equiv 4 \pmod{8}$
4. $x + 5 \equiv 4 \pmod{9}$
5. $2x \equiv 3 \pmod{6}$

Solving:

1. $x + 1 \equiv 3 \pmod{5}$

$x \equiv 2 \pmod{5}$

$x \in \{ \ldots\ldots., -13, -8, -3, 2, 7, 12, 17, \ldots\ldots.. \}$

$\therefore x \equiv 2 \pmod{5}$

$x = 2 + 5r$, for all $r \in Z$

2. $2x \equiv 4 \pmod{5}$

$2x - 4 \equiv 0 \pmod{5}$

$2(x - 2) \equiv 0 \pmod{5}$

$x - 2 \equiv 0 \pmod{5}$

$x \in \{ \ldots\ldots., -13, -8, -3, 2, 7, 12, 17, \ldots\ldots.. \}$

∴ $x \equiv 2 \pmod 5$
$x = 2 + 5r$, for all $r \in Z$

3. $2x \equiv 4 \pmod 8$
$2x - 4 \equiv 0 \pmod 8$
$2(x - 2) \equiv 0 \pmod 8$
$2(x - 2) = 0 + 8r$, for all $r \in Z$
$x - 2 = 4r$, for all $r \in Z$
$x = 2 + 4r$, for all $r \in Z$
$x \in \{ \ldots, -6, -2, 2, 6, 10, 14, \ldots \}$

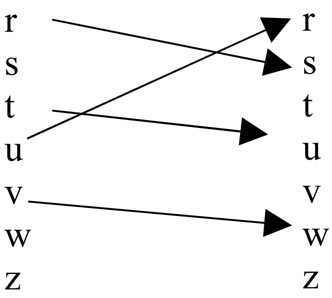
4. $x + 5 \equiv 4 \pmod 9$
$x \equiv -1 \pmod 9$
$x \equiv 8 \pmod 9$, where $9 + -1 = 8$
∴ $x = 8 + 9r$, for all $r \in Z$
$x \in \{ \ldots, -10, -1, 8, 17, 26, 35, \ldots \}$

5. $2x \equiv 3 \pmod 6$
$2x = 3 + 6r$ for some $r \in Z$
$2x - 6r$ divisible by 2
but 3 not divisible by 2
then Congruence has no relation.

6. $3x \equiv 3 \pmod 6$
$3x = 3 + 6r$ for some $r \in Z$
$1x = 2 + 6r$ divisible by 3
$x = 1 + 6r$ (mod 2)

Supposed:
Let A a finite set and let f: A → A be a permutation on A, then f defined equivalence relation R_f by
$x R_f y$ iff there exist $n > 0$, such that $y = f^n(x)$, where f^n is the function f composed n times.

Example:
Let A = { r, s, t, u, v, w, z } and let f: A → A defined by



Find:

1. R_f = { (r, s), (r, t), (r, u), (r, r),
(s, t), (s, u), (s, r), (s, s),
(t, u), (t, r), (t, s), (t, t),
(u, r), (u, s), (u, t), (u, u),
(v, w), (v, z), (v, v),
(w, z), (w, v), (w, w),
(z, v), (z, w), (z, z) }

r

1.1. $f(r) = f^1(r) = s$
1.2. $f \circ f(r) = f^2(r) = t$
1.3. $f \circ f \circ f(r) = f^3(r) = u$
1.4. $f \circ f \circ f \circ f(r) = f^4(r) = r$
1.5. $f \circ f \circ f \circ f \circ f(r) = f^5(r) = s$

s

1.1. $f(s) = f^1(s) = t$
1.2. $f \circ f(s) = f^2(s) = u$
1.3. $f \circ f \circ f(s) = f^3(s) = r$
1.4. $f \circ f \circ f \circ f(s) = f^4(s) = s$
1.5. $f \circ f \circ f \circ f \circ f(s) = f^5(s) = t$

t

1.1. $f(t) = f^1(t) = u$
1.2. $f \circ f(t) = f^2(t) = r$
1.3. $f \circ f \circ f(t) = f^3(t) = s$
1.4. $f \circ f \circ f \circ f(t) = f^4(t) = t$
1.5. $f \circ f \circ f \circ f \circ f(t) = f^5(t) = u$

u

1.1. $f(u) = f^1(u) = r$
1.2. $f \circ f(u) = f^2(u) = s$
1.3. $f \circ f \circ f(u) = f^3(u) = t$
1.4. $f \circ f \circ f \circ f(u) = f^4(u) = u$
1.5. $f \circ f \circ f \circ f \circ f(u) = f^5(u) = r$

v

1.1. $f(v) = f^1(v) = w$
1.2. $f \circ f(v) = f^2(v) = z$
1.3. $f \circ f \circ f(v) = f^3(v) = v$
1.4. $f \circ f \circ f \circ f(v) = f^4$

w

1.1. $f(w) = f^1(w) = z$
1.2. $f \circ f(w) = f^2(w) = v$
1.3. $f \circ f \circ f(w) = f^3(w) = w$

(v) = w

1.4. f ○ f ○ f ○ f (w) = f^4 (w) = z

z

1.1. f (z) = f^1 (z) = v

1.2. f ○ f (z) = f^2 (z) = w

1.3. f ○ f ○ f (z) = f^3 (z) = z

1.4. f ○ f ○ f ○ f (z) = f^4 (z) = v

NOTE:

f^0 (x) = x

f^1 (x) = f (x) = y

f^2 (x) = f [f (x)]

f^3 (x) = f (f [f (x)])

f^4 (x) = f [f (f [f (x)])]

f^5 (u) = f (f [f (f [f (x)])])