Department of Mathematics
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Discrete Mathematics

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CHAPTER FIVE:

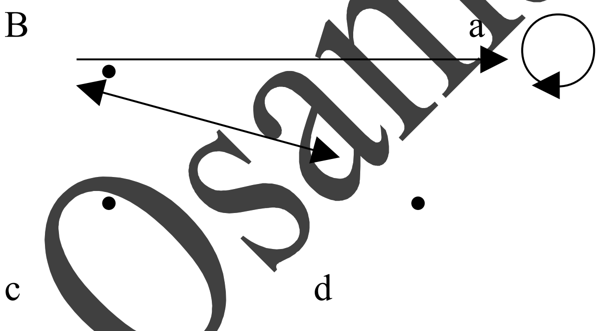
RELATIONS

SECTION 5.1:
Relations.

Definition:
Let A be a set, a subset R of A × A is called **a relation** on A
If pair (x, y) ∈ R, then we say that x related to y and write $_xR_y$ (x related y), otherwise we write $_x\not{R}_y$ (x NOT related y)

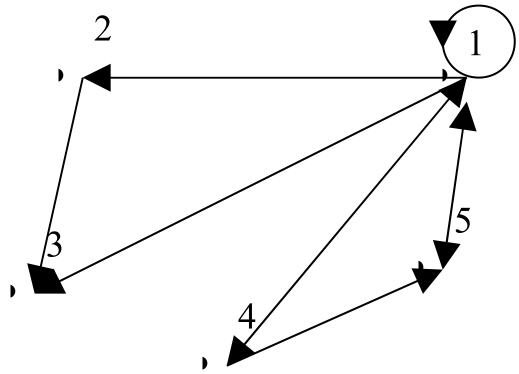
Example:
Let A = { a, b, c, d }, R = { (a, c), (a, d), (b, a), (a, a), (d, c), (d, d) } , then R is called a relation on A.
Is $_aR_b$? →since (a, b) ∉ R then $_a\not{R}_b$
Is $_bR_a$? →since (b, a) ∈ R then $_bR_a$
Is $_dR_c$? →since (d, c) ∈ R then $_dR_c$
Is $_dR_c$? →since (d, c) ∉ R then $_d\not{R}_c$

Example:
Let A = Z, and define a relation R on Z by (n, m) ∈ R, iff n – m ≥ 0
Is (1, 2) ∈ R ? → NO (1, 2) ∉ R (1-2= -1) then $_1\not{R}_2$
Is (2, 1) ∈ R ? → YES (2, 1) ∈ R (2-1= 1) then $_2R_1$
NOTE: if n ≥ m then (n, m) ∈ R

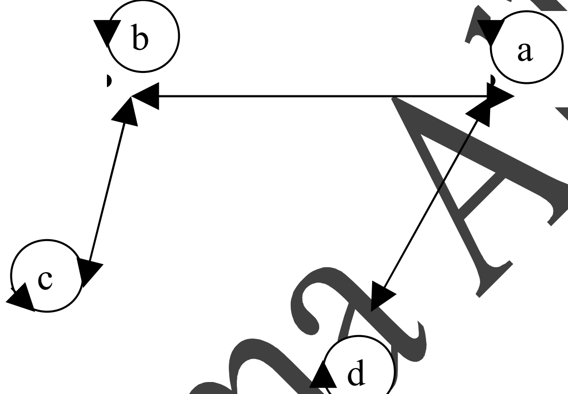


We can represent a relation on a finite set A with a diagram, this diagram is called a **directed graph** or **diagraph** each element in A is denoted by a **node** or **vertex** if (s, t) ∈ R, we connect s to t by an arrow is called **directed edge**.
If (s, s) ∈ R, then the arrow from s to s (to itself) is called a **loop** or **self loop**

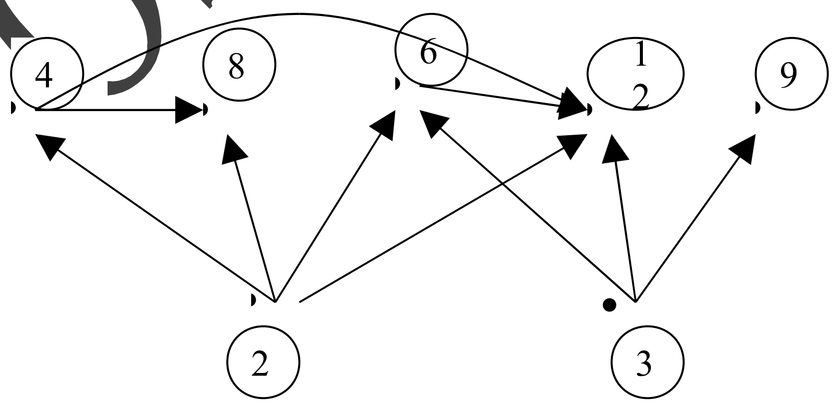
Example:
Let S = { 1, 2, 3, 4, 5 },
and let R = { (1, 1), (1, 2), (2, 3), (1, 3), (1, 4), (4, 5), (5, 1), (1, 5), (4, 1) }
draw a diagraph of R



Example:
draw a diagraph of the following A = { a, b, c, d },
R = { (a, a), (a, b), (b, a), (b, b), (c, b), (c, c), (b, c), (d, a), (d, d), (a, d) }

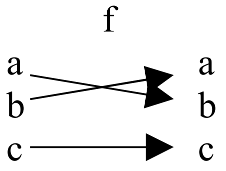


Example:
Let S = { 2, 3, 4, 6, 8, 9, 12 },
and let R be defined as (x, y) ∈ R iff x divides y
R = { (2, 2), (2, 4), (2, 6), (2, 8), (2, 12), (3, 3), (3, 6), (3, 9), (3, 12),
(4, 4), (4, 8), (4, 12),(6, 6), (6, 12), (8, 8), (9, 9), (12, 12) }
draw a diagraph of R

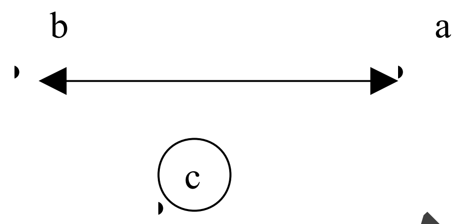


Remark:
Every function of the form f: A → A defines a relation R_f on A. By
(x, y) ∈ R_f iff f (x) = y

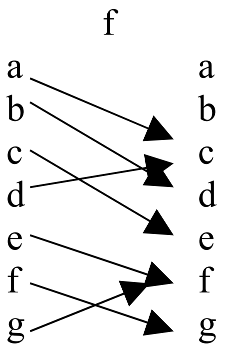
Example:
Let A = { a, b, c }, and f: A → A is defined by

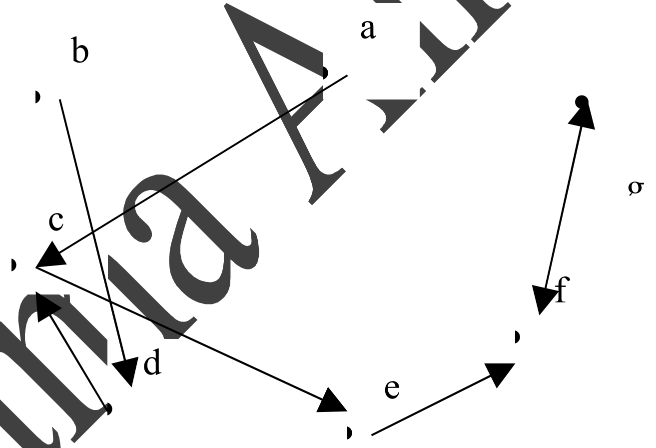


R_f = { (a, b), (b, a), (c, c) }



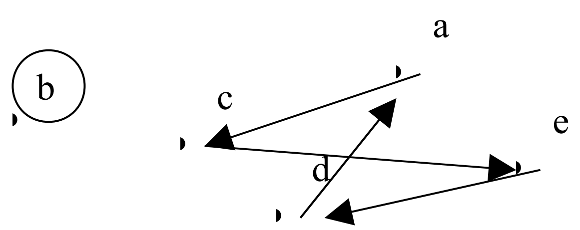
Question5:
Draw a diagraph of the relation R_f defined by the following function





Conversely:
Let R be a relation on a set A, if exactly one out-going arrow from each node, then we can use R to define a function f_R on A as follows f_R (a) = y iff (a, y) ∈ R

Example:
Let A = { a, b, c, d, e }, and let R be a relation defined diagraph
f_R : A → A defined by:
f_R (a) = c
f_R (b) = b
f_R (c) = e
f_R (d) = a
f_R (e) = d



Definition:
Suppose that R is a relation defined on A, then

1. a Relation (R) is **Reflexive** iff wherever $_xR_x$ for all $x \in A$.
   if $(x, x) \in R$, $\forall x \in A$, is called **Reflexive**
2. a Relation (R) is **Symmetric** iff wherever $_xR_y$ we have $_yR_x$.
   if $(x, y) \in R$, then $(y, x) \in R$, is called **Symmetric**
3. a Relation (R) is **Transitive** iff wherever $_xR_y$ and $_yR_z$, we have $_xR_z$.
   if $(x, y) \in R$, and $(y, z) \in R$, then $(x, z) \in R$ is called **Transitive**
4. a Relation (R) is **Irr-Reflexive** iff wherever $_xR_x$ for all $x \in A$.
   if $(x, x) \notin R$, $\forall x \in A$, is called **Irr-Reflexive**
5. a Relation (R) is **Anti-Symmetric** iff wherever $_xR_y$ and $_yR_x$, we have $x = y$.
   if $(x, y) \in R$, and $(y, x) \in R$, then $x = y$ is called **Anti-Symmetric**

Properties of a relation $R \subseteq A \times A$.

1. **Reflexive** if $\forall a \in A \ (a, a) \in R$
2. **Symmetric** if $\forall a \in A \ \forall b \in A, (a, b) \in R \rightarrow (b, a) \in R$
3. **Irr-Reflexive** if $\forall a \in A \ (a, a) \notin R$
4. **Anti-symmetric** if $\forall a \in A \ \forall b \in A, (a, b) \in R \wedge (b, a) \in R \rightarrow a = b$
5. **Transitive** if $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$

Compare a relation $R \subseteq A \times A$:

Range	Relation 1	Relation 2
$\forall x \in A$	**Reflexive**	**Irr-Reflexive**
	$_xR_x \in$ **A**	$_xR_x \notin$ **A**
$\forall x \in A$, $\forall y \in A$	**Symmetric**	**Anti-symmetric**
	$_xR_y \in$ **A,** $_yR_x \in$ **A**	$_xR_y \in$ **A,** $_yR_x \in$ **A (x=y)**

Example:
Determine whether the following relations reflexive, symmetric, transitive, antisymmetric , irreflexive
$A = \{ a, b, c, d \}$ $R = \{ (a, b), (b, d), (a, d), (d, a), (d, b), (b, a), (c, c) \}$

1. R is NOT reflexive ($(a, a) \notin R$)
2. R is symmetric

3. R is NOT transitive
(a, b) ∈ R and (b, a) ∈ R but (a, a) ∉ R
4. R is NOT Irr-reflexive ((c, c) ∈ R)
5. R is NOT anti-symmetric

Example:
A = { a, b, c, d, e } R = { (a, a), (a, b), (b, c), (a, c), (b, a), (c, a), (c, b), (d, e) }
1. R is NOT reflexive ((b, b) ∉ R)
2. R is NOT symmetric (d, e) ∈ R but (e, d) ∉ R
3. R is NOT transitive
(b, c) ∈ R and (c, b) ∈ R but (b, b) ∉ R
4. R is NOT irreflexive ((a, a) ∈ R)
5. R is NOT anti-symmetric (a, b) ∈ R and (b, a) ∈ R but a ≠ b

Example:
A = { 1, 2, 3, 4, 5 } and define a relation R on A by (x, y) ∈ R if $x - y \geq 0$

1. R is reflexive iff (x, x) ∈ R
iff $x - x \geq 0$
$0 \geq 0$
then: R is reflexive

2. R is symmetric (x, y) ∈ R and (y, x) ∈ R
$x - y \geq 0 \neq x - y \geq 0$
then: R is NOT symmetric

3. R is transitive iff
(x, y) ∈ R ∧ (y, z) ∈ R → (x, z) ∈ R
(x, y) ∈ R iff $x - y \geq 0$
(y, z) ∈ R iff $y - z \geq 0$
$x - z = x - y + y - z \geq 0$
then: (x, z) ∈ R
then: R is transitive

4. R is irreflexive iff (x, x) ∉ R for all x.
since (x, x) ∈ R for all x ∈ A
then: R is NOT irreflexive.

5. R is anti-symmetric iff (x, y) ∈ R ∧ (y, x) ∈ R but x = y
(a, b) ∈ R ∧ (b, c) ∈ R → (a, c) ∈ R
(x, y) ∈ R iff $x - y \geq 0$
(y, x) ∈ R iff $y - x \geq 0$
$x - y \geq 0 \wedge y - x \geq 0$

then: $x = y$
R is anti-symmetric

Example:

Let A = { a, b, c } R = { (a, a), (a, b), (b, c) }

	a	b	c
a	1	1	0
b	0	0	1
c	0	0	0

By General:

A relation R on a finite set A = { $a_1, a_2, a_3, a_4, \ldots\ldots\ldots, a_n$ } can be represented by a matrix M_R of 0's (zero's) and 1's (one's), by letting the entry (i_{th}) in the row and (j_{th}) column equal 1 if $a_i R a_j$ and equal 0 if $a_i \not R a_j$

Example:

Let A = { 1, 2, 3, 4, 5 }, and let

R = { (1, 1), (2, 2), (2, 1), (3, 3), (3, 2), (3, 1), (4, 4), (4, 3), (4, 2), (4, 1), (5, 5), (5, 4),(5, 3), (5, 2), (5, 1) }

Then M_R =

	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	1	1	1	0	0
4	1	1	1	1	0
5	1	1	1	1	1

Remark:

1. the relation R is reflexive iff the entries on the <u>diameter</u> are all 1's (one's)
2. the relation R is irreflexive iff the entries on the <u>diameter</u> are all 0's (zero's)

Example:

Let A = { a, b, c, d } and R = { (a, b), (b, d), (a, d), (d, a), (d, b), (b, a), (c, c) }

Then M_R =

	a	b	c	d
a	0	1	0	1
b	1	0	0	1
c	0	0	1	0
d	1	1	0	0

$x R y = 1$

$x \not R y = 0$

Every n × n matrix defined a relation R_M an a set { $a_1, a_2, a_3, a_4, \ldots\ldots\ldots, a_n$ } by $a_i R a_j$ iff the entry in the row (i) and column (j) equal zero

Example:

Let M =

	a_1	a_2	a_3
a_1	1	0	1
a_2	0	1	0
a_3	0	0	0

Let $A = \{ a_1, a_2, a_3\}$

$R_M = \{ (a_1, a_1), (a_1, a_3), (a_2, a_2) \}$