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Discrete Mathematics

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CHAPTER FOUR

COUNTING AND COUNTABILITY.

SECTION 4.1:
Counting Principles.

Definition:
Let A be a nonempty finite set. Let n be a positive integer, then cardinality (A) = $|A|$ = n iff there is a bijection from A to the set X_n = { 1, 2,. , n }, the cardinality of the empty set is zero (0).

Example:
Let A = { a, b, c } B = { {1}, {1, 2 } } C = Φ

1. $|A| = 3$
   let f: A → X_3 = { 1, 2, 3 } be a function defined by
   f
   a → 2
   b → 1 this function 1 -1 and onto (bijection)
   c → 3

2. $|B| = 2$
   let g: B → X_2 = { 1, 2 } be a function defined by
   g
   { 1 } → 1
   {1, 2 } → 2 this function 1 -1 and onto (bijection)

Proposition:
Let A and B be two nonempty finite sets, then $|A| = |B|$ iff there a bijection function form A to B.

Theorem:
The addition principles
Let A and B be two finite disjoint sets, $|A| = n$, $|B| = m$ then
$|A \cup B| = |A| + |B| = n + m$

Example:
Let A = { 1, 2, 3 } B = { 3, 4, 5 }
$|A| = 3$ $|B| = 3$
$A \cup B$ = { 1, 2, 3, 4, 5 }
Then $|A \cup B| = 5$
$|A \cup B| \neq |A| + |B| \neq A \cap B = \Phi$

Definition:
Let $A_1, A_2, A_3, \ldots A_n$ be a collection of sets, then if $A_i \cap A_j = \Phi$, for all $i \neq j$, then this collection called **mutually disjoint** .

Corollary:
If $A_1, A_2, A_3, \ldots A_n$ is mutually disjoint collection of sets, then.
$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$

Theorem:
If A a finite set, $|A| = n$ then $|P(A)| = 2^n$

Proposition:
Suppose that A is a subset of a finite universal set U, then
$|A^c| = |U| - |A|$

Example:
Let U = { 1, 2, 3, 4, 5, 6 } A = { 2, 4, 6 }
$|A^c| = |U| - |A|$
$= 6 - 3$
$= \mathbf{3}$

Theorem:
Suppose that A and B are two finite sets, then
$|A/B| = |A| - |A \cap B|$

Example:
Find the number of integer between 1 and 10 that are divisible by 3, 4, 6.
Solution:
Suppose A = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }
The # of elements divisible by 3 = { 3, 6, 9 } = 3
The # of elements divisible by 4 = { 4, 8 } = 2
The # of elements divisible by 6 = { 6 } = 1

Clarification:
10 = **3** + **3** + **3** + 1
10 = **4** + **4** + 2
10 = **6** + 4
Other solution:
$\left[\frac{10}{3}\right] = 3$ $\left[\frac{10}{4}\right] = 2$ $\left[\frac{10}{6}\right] = 1$

Example:

Find the number of integer between 1 and 100 that are not divisible by 3.

Solution:

The # of elements divisible by 3 = $\left\lfloor \frac{100}{3} \right\rfloor$ = 33

Let A be the set integers divisible by 3

$|A^c| = |U| - |A|$

$= 100 - 33$

$= 67$

Example:

Find the number of integer between 1 and 100 that are divisible by 3 but not divisible by 5.

Solution:

Numbers that are divisible by 3 and 5 is the number 15

(the first number * second number)

The # of elements divisible by 15 = $|A \cap B| = \left\lfloor \frac{100}{15} \right\rfloor = 6$

The # of elements divisible by 3 = $|A| = \left\lfloor \frac{100}{3} \right\rfloor = 33$

The # of elements divisible by 3 and not divisible by 5

$|A/B| = |A| - |A \cap B|$

$= \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor$

$= 33 - 6$

$= 27$

The inclusion and Exclusion Principle

Theorem:

Suppose that A and B are two finite sets then

$|A \cup B| = |A| + |B| - |A \cap B|$

Example:

Let $|A| = 100$, $|B| = 150$, $|A \cup B| = 200$, Find $|A \cap B|$

$|A \cup B| = |A| + |B| - |A \cap B|$

$200 = 100 + 150 - |A \cap B|$

$200 = 250 - |A \cap B|$

$|A \cap B| = 50$

Theorem:

Suppose that A , B and C are three finite sets then

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Corollary:

$$|A_1 \cup A_2 \cup \ldots\ldots\ldots\ldots\ldots \cup A_n| = |A_1| + |A_2| + \ldots\ldots + |A_n| - |A_1 \cap A_2| - |A_{n-1} \cap A_n|$$

$$+ |A_1 \cap A_2 \cap A_3| + (-1)^{n+1} |A_1 \cap A_2 \cap A_n|$$

OR

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=0}^{n} |A_k| - \sum_{\substack{i,j \\ i \neq j}}^{n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Example:

Determine how many integers between 1 and 1235 are divisible by 2 or 3

Solution:

Let:

A = the set of all integers between 1 and 1235 are divisible by 2

B = the set of all integers between 1 and 1235 are divisible by 3

$|A| = \left[\frac{1235}{2}\right] = 617$ $|B| = \left[\frac{1235}{3}\right] = 411$

$|A \cap B| = \left[\frac{1235}{2 \times 3}\right] = 205$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 617 + 411 – 205$

$= 823$

Example:

Find the numbers of integers between 1 and 1000 that are divisible by 2,3 and 5

Solution:
Let:
A = the set of all integers between 1 and 1000 are divisible by 2
B = the set of all integers between 1 and 1000 are divisible by 3
C = the set of all integers between 1 and 1000 are divisible by 5

$|A| = \left[\frac{1000}{2}\right] = 500$

$|B| = \left[\frac{1000}{3}\right] = 333$

$|C| = \left[\frac{1000}{5}\right] = 200$

$|A \cap B| = \left[\frac{1000}{2\times 3}\right] = 166$

$|A \cap C| = \left[\frac{1000}{2\times 5}\right] = 100$

$|B \cap C| = \left[\frac{1000}{3\times 5}\right] = 66$

$|A \cap B \cap C| = \left[\frac{1000}{2\times 3\times 5}\right] = 33$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$= 500 + 333 + 200 - 166 - 100 - 66 + 33$
$= 846$

The multiplication Principle
Theorem:
Suppose that $|A| = n$, $|B| = m$, then $|A \times B| = n \times m$
In general:

$$|A_1 \times A_2 \times A_3 \ldots\ldots\ldots\ldots \times A_n| =$$
$$|A_1| \times |A_2| \times |A_3| \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \times |A_n|$$

Example:
How many six place licenses plate are possible if the first three places must be letters and the next three are digit.
Sol:
$= 26 \times 25 \times 24 \times 10 \times 9 \times 8$
$= 11232000$

Example:
A code word containing 0 and 1 only can be of length 3, 4 or 5
How many such code word are possible
A: 2^3
B: 2^4
C: 2^5
We sets are disjoint then
$|A \cup B \cup C| = |A| + |B| + |C|$
$= 2^3 + 2^4 + 2^5$
$= 8 + 16 + 32$
$= 56$

Example:
Determine how many words of length 7 can be formed format the set of letters { a, b, c, d, e } if the first letter is a OR the last letter is b.
Solution:
A: the set word with first letter is a
B: the set word with first letter is b
$|A| = 1 \times 5^6$
$|B| = 5^6 \times 1$
$|A \cap B| = 1 \times 5^5 \times 1$
$|A \cup B| = |A| + |B| - |A \cap B|$
$= 5^6 + 5^6 - 5^5$
$= 15625 + 15625 - 3125$
$= 28125$

Question 12 / page 141:
How many integers between 999 and 9999 either begin or end 3

Solution:
Let A : the set of integers between 999 and 9999 begin with 3
Let B : the set of integers between 999 and 9999 ends with 3
$|A| = 1 \times 10^3$ (begin with 3→ 9:**1**, 9:**10**, 9:**10**, 9:**10**)
$|B| = 9 \times 10 \times 10 \times 1$ (end with 3→ 9:**9**, 9:**10**, 9:**10**, 9:**1**)
$|A \cap B| = 1 \times 10^2 \times 1$ (begin and end with 3→ 9:**1**, **9**:**10**, 9:**10**, 9:**1**)
$|A \cup B| = |A| + |B| - |A \cap B|$
$= 10^3 + 900 - 100$
$= 1000 + 900 - 100$
$= 1800$