Department of Mathematics
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Discrete Mathematics

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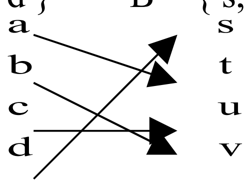
CHAPTER THREE:

FUNCTIONS.

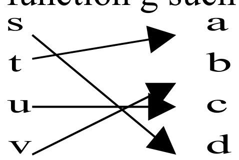
SECTION 3.4:
Inverse functions.

The inverse of a function $f : A \to B$ is the relation define a bijection relation $f^{-1} : B \to A$ is the relation given by
$f^{-1}(x) = y \leftrightarrow f(y) = x$.

Clarification:
Let A = { a, b, c, d }   B = { s, t, u, v }

Let f: A → B by 

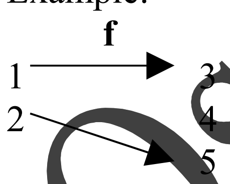
I want to find a function g such that $g \circ f(x) = x$

g : B → A = 

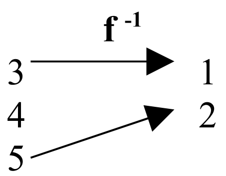
if $x \in A$, $y \in B$ $f(x) = y \to g(y) = x$, then
this function is called the **inverse function**

Definitions:
Suppose that f: A → B is bijection we define the function $f^{-1} : B \to A$, called the **inverse function (f)** by the following rule:
if $z \in B$, then $f^{-1}(z) = x$, iff $f(x) = z$

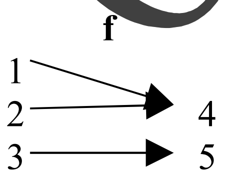
Example:



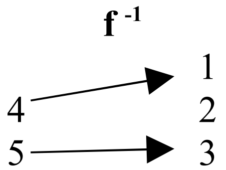
1- 1 (not onto)



not function



onto (not 1-1)



not inverse function

Proposition:
Suppose that $f: A \rightarrow B$ is a bijection (bijective) $f^{-1} = B \rightarrow A$ is its inverse, then

1. for all $x \in A$, we have $f^{-1} \circ f(x) = x$
2. for all $x \in B$, we have $f \circ f^{-1}(x) = x$

Proof:

1. for all $x \in A$, we have $f^{-1} \circ f(x) = x$

$f^{-1} \circ f(x) = x, \ \forall x \in A \ \ f: A \rightarrow B$

let $f(x) = z \in B$

$f^{-1}(z) = x$

$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(z) = x$

2. $f^{-1}(x) = x \ \ \forall x \in B$

$f^{-1}: B \rightarrow A$

let $f^{-1}(x) = z \in B$

$f : A \rightarrow B \qquad f(z) = x$

$f \circ f^{-1}(x) = f(f^{-1}(x)) = f(z) = x$

NOTE:
$f: A \rightarrow B$, $f^{-1} : B \rightarrow A$, that is to say

1. $f \circ f^{-1} = id_B$
2. $f^{-1} \circ f = id_A$

Example:
Let $f: R \rightarrow R$ be defined by $f(x) = 3x - 1$, find f^{-1} if possible.
Solution:

1. the function is bijection
2. $f^{-1} : R \rightarrow R$ is defined by $f^{-1}(y) = \frac{y+1}{3}$

let $f^{-1}(y) = x$, then $f(x) = y \quad 3x - 1 = y$

$$3x = y + 1$$

$$x = \frac{y+1}{3}$$

$$f^{-1}(x) = \frac{x+1}{3}$$

Example:
let $f: R \rightarrow R$ be defined by $f(x) = [x]$, find f^{-1} if possible.
Solution:
This function is not 1 – 1
$x = 0.1$
$y = 0.3$
then $f(x) = f(y) = 0$
So, the function f has no inverse.

Example:

Let h: E → Z, where **E** is the set of all even integer be defined by $h(x) = \frac{x}{2}$,
find f^{-1} if possible.
Solution:

1. the function is injective

let h (x) = h (y) (want x = y)

$\frac{x}{2} = \frac{y}{2}$

x = y

2. the function is surjective

let $y \in Z$ (want $\exists x \in E : h(x) = y$)

let h (x) = y $\quad \frac{x}{2} = y$

$x = 2y$

let $x = 2y \in E$, then h (x) = y

let $h^{-1}(y) = x$, then h (x) = y

$\frac{x}{2} = y$

$x = 2y$

then h^{-1} is defined by $h^{-1}(x) = 2x$

Example:

Let B = { x : x ≥ 1 or x < 0 } and defined by

$$f(x) = \begin{cases} x + 1 & x \geq 0 \\ 2x & x < 0 \end{cases}$$

find f^{-1} if possible.
Solution:

1. the function is bijection
2. $f^{-1} : B \to R$ is defined by let $f^{-1}(y) = x$

i. $x \geq 0$ $\quad x + 1 = y$

$x = y - 1 \quad y \geq 1$

ii. $x < 0$ $\quad 2x = y$

$x = \frac{y}{2} \quad y < 0$

$$f^{-1}(x) = \begin{cases} y - 1 & y \geq 1 \\ \frac{y}{2} & y < 0 \end{cases}$$

y ≥ 1 : from define B
y < 0 : from define B

Example:
let f : N → Z and defined by

$$f(n) = \begin{cases} \frac{n}{2} & n, \text{ even} \\ \frac{1-n}{2} & n, \text{ odd} \end{cases}$$

find f^{-1} if possible.

Solution:
f^{-1} : Z → N is defined by
let $f^{-1}(m) = n \rightarrow f(n) = m$

i. n is even $\frac{n}{2} = m$

$n = 2m$

$f^{-1}(m) = 2m, \quad m > 0$

ii. n is odd $\frac{1-n}{2} = m$

$1-n = 2m$

$n = 1 - 2m, \quad m \leq 0$

$n > 0 \quad f^{-1}(m) = 1- 2m \quad m \leq 0$

$- n < 0 \quad \frac{1-n}{2} < \frac{1}{2}$

$m < \frac{1}{2}$

$m \leq 0$

then

$$f^{-1}(n) = \begin{cases} 2n & n > 0 \\ 1-2n & n \leq 0 \end{cases}$$

Question:
Find the inverse of each of the following bijective (bijection)

1. f: R → R defined by $f(x) = \frac{1-2x}{3}$
2. f: R → R defined by

$$f(x) = \begin{cases} x^2 & x \geq 2 \\ x+2 & x < 2 \end{cases}$$

solution:

1. f: R → R defined by $f(x) = \frac{1-2x}{3}$

$f^{-1}(y) = x$ $\quad$ $f(x) = y \rightarrow \frac{1-2x}{3} = y$

$$1 - 2x = 3y$$
$$2x = 1 - 3y$$
$$x = \frac{1-3y}{2}$$

$f^{-1}(x) = \frac{1-3x}{2}$

2. f: R → R defined by

$$f(x) = \begin{cases} x^2 & x \geq 2 \\ x+2 & x < 2 \end{cases}$$

f^{-1}: R → R $\quad$ $f^{-1}(y) = x \rightarrow f(x) = y$

i. $x \geq 2$, $\quad$ $f(x) = x^2 = y$

$$|x| = \sqrt{y}$$
$$x = \pm\sqrt{y}$$

x is positive $\quad$ then $x = \sqrt{y}$

$f(y) = \sqrt{y} = f(y) \geq 4$

ii. $x < 2$, $\quad$ $f(x) = x + 2 = y$

$$x = y - 2$$

$f(y) = y - 2$, $y < 4$

$$f^{-1}(x) = \begin{cases} \sqrt{x} & x \geq 4 \\ x-2 & x < 4 \end{cases}$$

NOTE:
Solving for an inverse function algebraically is a three steps process:

1. set the function equally y
2. swap the x and y variables
3. solve for y