Department of Mathematics
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Discrete Mathematics

Yarmouk University

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CHAPTER THREE:

FUNCTIONS.

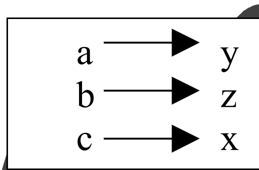
Section 3.3
Composition of functions

Suppose there are two functions $f : A \rightarrow B$ and $g : B \rightarrow C$. The composition function $g \circ f : A \rightarrow C$ is defined by $g \circ f\,(a) = c \leftrightarrow f\,(a) = b \wedge g(b) = c$. In particular when $A = B = C$ this definition coincides with that of arbitrary relations on A.

NOTE:
To define **g ○ f** must be the Domain (g) **itself** the CoDomain (f).

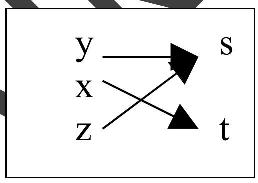
Example:
Let A = { a, b , c } B = { x, y, z } C = { s, t } and defined:

1. f: A → B by
   f (a) = y f (b) = z f (c) = x



2. g: B → C by

   g (y) = s g (z) = s g (x) = t



3. g (f (a)) = g (y) = s
   g ○ f (x) = g (f (x))

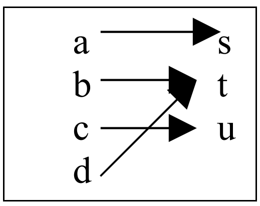
   g (f (a)) = g (y) = s
   g (f (b)) = g (z) = s
   g (f (c)) = g (x) = t

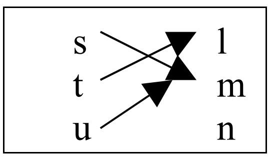
f: A → B
g: B → C
g ○ f: A → C

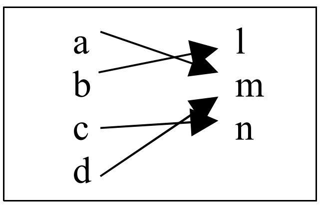
we can define a new function h: A → C by h (x) = g (f(x)), this function is called the **composition of function** (f) with (g), and is denoted by **(g circle f) (g ○ f)** and defined by g ○ f(x) = g (f (x))

Example:
let f, g be defined by the diagrams

f g g ○ f

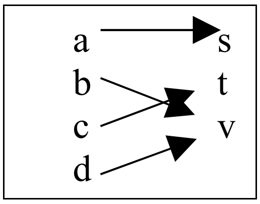


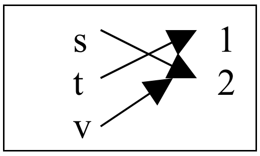


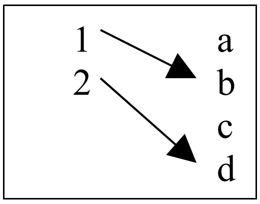


Question 1:

f g h







Find:

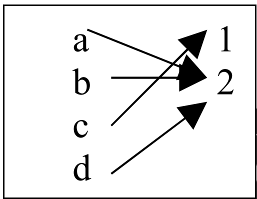
1. f ○ g if possible?
2. g ○ f if possible?
3. f ○ h if possible?
4. h ○ f if possible?
5. h ○ g if possible?
6. g ○ h if possible?

- f ○ g if possible?

  Not defined f (g(x))

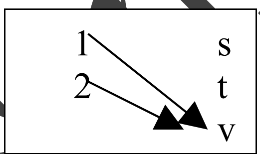
- g ○ f if possible?

  = g (f(x))



- f ○ h if possible?

  = f (g(x))

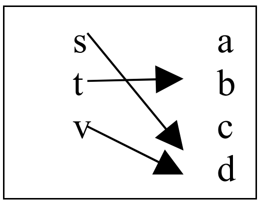


- h ○ f if possible?

  Not defined h (f(x))

- h ○ g if possible?

  = f (g(x))



- g ○ h if possible?

Not defined g (h(x))

Note:
f: Dom(f) → CoDom(f)
g: Dom(g) → CoDom (g)
and ($g \circ f = f \circ g$) IF [Dom(g) = CoDom(f) = Dom(f) = CoDom (g)]

Example:
suppose f: R → R defined by $f(x) = x^2 + 3$.
g: R → R defined by $g(x) = 4x - 5$
Find $f \circ g$ and $g \circ f$, if possible ?

$$f \circ g = f(g(x)) = [g(x)]^2 + 3 = (4x - 5)^2 + 3 = 16x^2 + 25 - 40x + 3 = 16x^2 - 40x + 28$$

$$g \circ f = g(f(x)) = 4[f(x)] - 5 = 4(x^2 + 3) - 5 = (4x^2 + 12) - 5 = 4x^2 + 7$$

Example:

Let f: R → R defined by $f(x) = \begin{cases} x^2 + 1 & x > 0 \\ 2x + 1 & x \le 0 \end{cases}$

Let g: R → R defined by $g(x) = \begin{cases} x^3 & x > 0 \\ 3x - 7 & x \le 0 \end{cases}$

$$f \circ g = f(g(x)) = \begin{cases} [g(x)]^2 + 1 & g(x) > 0 \\ 2[g(x)] + 1 & g(x) \le 0 \end{cases}$$

$$= \begin{cases} (x^3)^2 + 1 & x > 0 \\ 2(3x - 7) + 1 & x \le 0 \end{cases}$$

$$g \circ f = g(f(x)) = \begin{cases} [f(x)]^3 & f(x) > 0 \\ 3[f(x)] - 7 & f(x) \le 0 \end{cases}$$

$$= \begin{cases} (2x + 1)^3 & x > 0 \\ (2x + 1)^3 & \frac{-1}{2} < x \le 0 \\ 3(2x + 1) - 7 & x \le \frac{-1}{2} \end{cases}$$

Theorem:
Suppose that $f: A \to B$ and $g: B \to C$, then

i. if f and g are both injective then $g \circ f$ is injective
ii. if f and g are both surjective then $g \circ f$ is surjective

Proof:

i. if f and g are both injective then $g \circ f$ is injective

$g \circ f : A \to C$
let $x, y \in A$ with $x \neq y$ (want $g \circ f\ (x) \neq g \circ f\ (y)$)
$\Rightarrow g \circ f\ (x) = g(f(x))$
$\Rightarrow g \circ f\ (y) = g(f(y))$
$x, y \in A$ with $x \neq y \to$ the function is $1 - 1$ then $f(x) \neq f(y)$
$f(x), f(y) \in B$ with $f(x) \neq f(y) \to 1 - 1$ then $g(f(x)), g(f(y)) \to g(f(x)) \neq g(f(y))$

ii. if f and g are both surjective then $g \circ f$ is surjective

let $y \in C$ (want $\exists x \in A : g \circ f\ (x) = y$)
$g: B \to C$
$y \in C$ and g surjective $\to \exists z \in B$ such that $g(z) = y$
$f: A \to B$
$z \in B$ and f surjective $\to \exists x \in A$ such that $f(x) = z$
then:
$g \circ f\ (x) = g(f(x)) = g(z) = y$

Corollary
suppose that $f: A \to B$ and $g: B \to C$ bijective functions, then $g \circ f : A \to C$ is bijective function.

Remark:
It is not true that if $g \circ f$ is surjective then g and f are both injective
Example: