Department of Mathematics
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Discrete Mathematics

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CHAPTER THREE:

FUNCTIONS.

SECTION 3.2:
Injective and surjective functions.

Definition:

i. A function f: A → B is called **injective (1 -1) (one to one)** if, wherever x, y ∈ A, with $x \neq y$, then $f(x) \neq f(y)$

ii. A function f: A → B is called **surjective (onto) (comprehensive)** if,
for all y∈ B, there exist x ∈ A such that $f(x) = y$

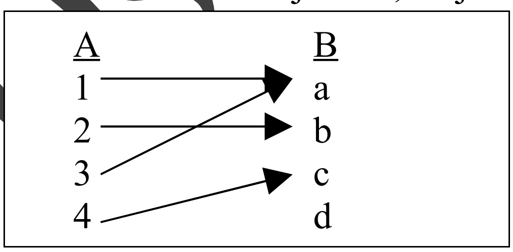
iii. A function that is both injective (1 – 1) and surjective (onto) is called **bijective** or **one to one correspondence**

Properties of a function f : A → B.
1) f is **one-to-one** or **an injection** if : $f(x) = f(y) \rightarrow x = y$.
2) f is **onto** or **a surjection** if : $f(A) = B$.
3) f is **a bijection** if both one-to-one and onto.

Example: All the following are functions define by f : A → B.

1. A = {1, 2, 3}, B ={x, y, z, w}, f = {(1,y), (2,z), (3,w)}
   is **one-to-one** but not onto.
2. A = {1, 2, 3}, B ={x, y, z, w}, f = {(1,y), (2,w), (3,w)}
   is neither one-to-one nor onto.
3. A = {1, 2, 3}, B ={x, y, z}, f = {(1,y), (2,z), (3,x)}
   is both **one-to-one** and **onto (bijective)**.
4. A = {1, 2, 3, 4}, B ={x, y, z}, f = {(1,y), (2,z), (3,y), (4,x)}
   is **onto** but not one-to-one.

Clarification for define Injective, surjective and bijective functions :

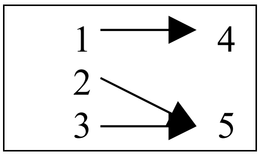


NOT one to one (1 - 1)
For each element in group A have one element in group B
NOT onto (comprehensive)
For each element in group A have all elements in group B, and every elements in group B have at least one element in group A.

Example:

Let A = { 1, 2, 3 } B = { 4, 5 }

f: A → B defined by



1. x, y ∈ A, x ≠ y → f (x) ≠ f(y)
   2, 3 ∈ A, but f (2) = f(3) = 5
   Then NOT one to one (1 – 1)

2. ∀y (CoDom A) ∈ B, ∃x ∈ A such that f (x) = y.
   4 ∈ B, 1∈ A and f (1) = 4 onto
   5 ∈ B, 2∈ A and f (2) = 5 onto

3. f: [2 , -2] → [0 , 5] defined by $f(x) = x^2 + 1$.
   $f(x) = x^2 + 1$
   $f(2) = 2^2 + 1$
   $= 5$
   $f(x) = x^2 + 1$
   $f(-2) = -2^2 + 1$
   $= 5$
   x, y ∈ A, x ≠ y → f (x) ≠ f(y)
   2, -2 ∈ A, 2 ≠ -2 → f (2) = f(-2)
   NOT one to one

Example:

A = { a, b, c } B = { s, t, u, v }

Let f: A → B defined by

f (a) = s f (b) = t f (c) = u

x, y ∈ A, x ≠ y → f (x) ≠ f(y)

a, b, c ∈ A, a ≠ b≠ c → f (a) ≠ f(b) ≠ f(c)

(1 – 1) injective.

v ∈ B but not ∃ x ∈ A such that f (x) = v

NOT onto

NOTE:

Suppose A, B are sets, and | A | = 3, | B | = 2, then:

1. If | A | > | B |, then any function f: A → B cannot be **1 – 1** or **injective**
2. If | A | < | B |, then any function f: A → B cannot be **onto** or **surjective**

Definition:

A function that is both injective (1 – 1) and surjective (onto) is called **bijective** or **one to one correspondence**

Example:
Let f: { 1, 2 } → { a, b } be defined by f (1) = a, f (2) = b
x, y ∈ A, x ≠ y → f (x) ≠ f(y)
the function is (1 – 1)
∀y (CoDom A) ∈ B, ∃x ∈ A such that f (x) = y.
the function is (onto)
Then is (1 – 1) and (onto) called **bijective**.

Example:
Let f: { 1, 2, 3 } → { a, b, c } be defined by f (1) = a, f (2) = a, f (3) = b
x, y ∈ A, x ≠ y → f (x) ≠ f(y)
1, 2 ∈ A, 1 ≠ 2 → f (1) = f(2)
the function is NOT (1 – 1)
∀y (CoDom A) ∈ B, ∃x ∈ A such that f (x) = y.
c ∈ B but not ∃ x ∈ A such that f (x) = c
the function is NOT (onto)
Then is NOT (1 – 1 and onto)

Example:
Suppose f: R → R defined by f (x) = x + 1
1. Is f injective (one to one) (1 – 1) ? **yes**
2. Is f surjective (onto) ? **yes**
3. Is f bijective? Then **yes**

Solution:

1. Is f injective (one to one) (1 – 1) ?
   f (x) = x + 1 ≠ y + 1 = f (y)
2. Is f surjective (onto) ?
   let y ∈ R, want ∃x : f (x) = y → f (y -1) = (y -1) + 1 = y
   let x = y – 1 ∈ R f (x) = f (y -1) = (y -1) + 1 = y

f (x) = y
x + 1 = y
x = y - 1

3. Is f bijective?
   the function is (1 – 1)
   the function is (onto)
   Then **yes**

Example:
Let f: R → R defined by f (x) = 3x - 1
1. Is f injective (one to one) (1 – 1) ? **yes**
2. Is f surjective (onto) ? **yes**
3. Is f bijective? Then **yes**

Solution:
1. Is f injective (one to one) (1 – 1) ?
suppose f (x) = f (y) (want x = y)
3x – 1 = 3y – 1
3x = 3y
x = y
2. Is f surjective (onto) ?
let y ∈ R, (want ∃x such that f (x) = y)

$$f\left(\frac{y+1}{3}\right) = 3\left(\frac{y+1}{3}\right) - 1 = y$$

$$= 3\left(\frac{y+1}{3}\right) \quad = y + 1$$

$$= \left(\frac{y+1}{3}\right) \quad = \frac{y+1}{3}$$

f (x) = y
3x - 1 = y
3x = y + 1
$x = \frac{y+1}{3}$

then $\frac{y+1}{3} \in R$ and $f\left(\frac{y+1}{3}\right) = y$

3. Is f bijective?
the function is (1 – 1)
the function is (onto)
Then **yes**

Example:
Let f: N → N defined by f (n) = 2^n
1. Is f injective (one to one) (1 – 1) ? **yes**
2. Is f surjective (onto) ? **No**
3. Is f bijective? Then **No**

1. Suppose f (n) = f (m) (want n=m)
$2^n = 2^m$ → n = m
2. 3 ∈ N but $3 \neq 2^n$ NO ∃n ∈ N such that f (n) = 2^n = 3

3. Is f bijective?
the function is (1 – 1)
the function is NOT (onto)
Then **Not** bijective

NOTE:
The function 1 -1 definition by If $x \neq y \rightarrow f(x) \neq f(y)$
then $f(x) = f(y) \rightarrow x = y$
then $x \neq y \rightarrow f(x) = f(y)$ [contradiction rule]

Example:
let f: R → R defined by $f(x) = \frac{2x - 1}{3}$
1. Is f injective (one to one) (1 – 1) ? **yes**
2. Is f surjective (onto) ? **yes**
3. Is f bijective? Then **yes**

1. Is f injective (one to one) (1 – 1) ?
suppose $f(x) = f(y)$

$$\frac{2x - 1}{3} = \frac{2y - 1}{3}$$

$2x - 1 = 2y - 1$
$2x = 2y$
$x = y$
Then Function is (1 – 1)

2. Is f surjective (onto) ?

$f\left(\frac{3y+1}{2}\right) = y$!!?

$$\frac{2\,\frac{3y+1}{2} - 1}{3} = y$$

$$\frac{3y}{3} = y$$

$y = y$

$f(x) = y$

$$\frac{2x - 1}{3} = y$$

$2x - 1 = 3y$
$2x = 3y + 1$

$$x = \frac{3y + 1}{2}$$

3. Is f bijective?
the function is (1 – 1), the function is (onto), Then **yes**

Example:
let f: Z → Z defined by $f(n) = 3n + 1$
is f one to one?
Suppose $f(n) = f(m)$ (want $n = m$)
$3n + 1 = 3m + 1$
$3n = 3m$
$n = m$
then one to one (1 – 1)
is f onto?
Counter example:
$2 \in Z$ but no $\exists\, n \in Z$ such that $f(n) = 3n + 1 = 2$

$f(n) = m$
$3n + 1 = m$
$3n = m - 1$

$$n = \frac{m - 1}{3}$$

no $\exists$ n $f(n) = 2$

Example:

Let f: R → R^+ defined by f (x) = $x^2 + 2$
1. Is f injective (one to one) (1 – 1) ? **no**
2. Is f surjective (onto) ? **no**
3. Is f bijective? Then **no**
Solution:
1. Is f injective (one to one) (1 – 1) ?

$2 \neq -2$ but $f(2) = 2^2 + 2 = -2^2 + 2 = f(-2)$
Not injective (1 – 1)
2. Is f surjective (onto) ?
$1 \in R^+$ but no $\exists x \in R$ such that $x^2 + 2 = 1$
Not surjective
3. Then not bijective

Example:
Let f: R → R defined by $f(x) = \begin{cases} x^2 - 2x + 5 & x > 1 \\ x - 1 & x \leq 1 \end{cases}$
1. Is f injective (one to one) (1 – 1) ? **yes**
2. Is f surjective (onto) ? **Yes**
3. Is f bijective? Then **Yes**
Case1:
$x, y \leq 1$ suppose f (x) = f (y) (want x = y)
$x - 1 = y - 1$
$x = y$
Case 2:
$x, y > 1$ Suppose f (x) = f (y) (want x = y)
$x^2 - 2x + 5 = y^2 - 2y + 5$
$x^2 - 2x = y^2 - 2y$
$x^2 - 2x - y^2 + 2y = 0$
$x^2 - y^2 - 2x + 2y = 0$
$(x^2 - y^2) - 2(x - y) = 0$
$(x + y)[(x + y) - 2] = 0$
$(x + y) - 2 = 0$
$x + y = 2$
$x = 2 - y$, and $y = 2 - x$
Case 3:
$y \leq 1$ and $x > 1$ $x \neq y$, f (x) = f (y) (want contradiction)
$x^2 - 2x + 5 = y - 1$
$x^2 - 2x + 5 - y + 1 = 0$
$(x - 1)^2 + 5 - y = 0$
$(x - 1)^2 = y - 5$