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Discrete Mathematics

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Chapter Two

SET THEORY

Section 2.9
Proof by Induction

Proof by Mathematical Induction:
To prove a proposition in the form $\forall n\ P(n)$ where n is a natural number, it suffices to prove P(1) and P(n) → P(n+1).

Example:
Prove the following formula for all natural numbers n.
$1 + 3 + 5 + 7 + 9 + \ldots\ldots\ldots + (2n – 1) = n^2$
Solution:
Let P(n): $1 + 3 + 5 + 7 + 9 + ... + (2n – 1) = n^2$
We shall prove $\forall n\ P(n)$ in two steps:

1. P(1): $2n – 1 = 1 \rightarrow n^2 = 1^2$ so this proposition is true.
   P(2): $(2n – 1)+(2n – 1) =1+ 3 \rightarrow n^2 = 2^2$ so this proposition is true.
   P(3): $(2n – 1) +(2n – 1) +(2n – 1) =1+3+ 5 \rightarrow n^2 = 3^2$ so this proposition is true.
   …………
   P(k): $\sum_{x=1}^{k} 2x - 1 = k^2$ so this proposition is true.

2. P(n): $1 + 3 + 5 + 7 + 9 + ... + (2n – 1) = n^2$
   If **2n-1** is true then **2n+1** is true proposition.
   $\rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2n – 1) + (2n + 1) = n^2 + (2n + 1)$
   $\rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2n – 1) + (2n + 1) = \mathbf{(n + 1)^2}$
   We want to prove the formula is $(n + 1)^2$
   $\rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2k – 1) + (2k + 1) = k^2 + (2k + 1)$
   $\rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2k – 1) + (2k + 1) = k^2 + 2k + 1$
   $\rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2k – 1) + (2k + 1) = \mathbf{(k + 1)^2}$
   $\rightarrow \therefore$ P(k+1) is true

Steps of proof by mathematical induction:
1. n = 0
2. P(k) $\Rightarrow$ P (k+1) for any K

$\therefore$ P (n) $n \geq 0$ is true

- Mathematical Induction is a method to prove that a statement or a proposition of the form $\forall n\ P(n)$ is true where n is positive integer.

Method:
A proposition P (n) is true for $n \geq 1$ if
1. P (1) is true
2. P (k) $\Rightarrow$ P (k+1) is true for every k in N

Example:
Use mathematics induction to prove

$$\underbrace{1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + n}_{P(n)} = \frac{n(n+1)}{2}$$

Proof:

1. for n = 1

$P(1) = \frac{1(1+1)}{2} = 1$

$\therefore$ the proposition is true for n = 1

2. suppose that the statement is true for n = k

$1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + k = \frac{k(k+1)}{2}$

We want to prove:

$1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + k + (k+1) = \frac{K+1(k+2)}{2}$

3. For n = (k + 1)

$$P(k) = 1 + 2 + 3 + \ldots\ldots\ldots + k + (k+1) = \frac{K(k+1)}{2} + (k+1)$$

$$= \frac{K(k+1)}{2} + \frac{(k+1)}{1}$$

$$= \frac{K(k+1)}{2} + \frac{2(k+1)}{2 \times 1}$$

$$= \frac{K(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{K(k+1) + 2(k+1)}{2}$$

(k+1) : common factor

$$= \frac{(k+1)\quad(k+2)}{2}$$

$\therefore$ the proposition is true for n = k
From 1, 2 and 3:

$1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

Example:
Proof by induction

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + r^n = \frac{1-r^{n+1}}{1-r}$$

$n \geq 0$ for any real number.
Proof:

1. for $n = 0$

$$P(0) = 1 \rightarrow \quad \frac{1-r^{n+1}}{1-r} = \frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1$$

$\therefore$ the proposition is true for $n = 0$

$$P(k) = \frac{1-r^{k+1}}{1-r} \rightarrow \quad \frac{1-r^{n+1}}{1-r} = \frac{1-r^{k+1}}{1-r}$$

2. suppose the proposition is true for $n = k$

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + r^k = \frac{1-r^{k+1}}{1-r}$$

3. For $n = k + 1$

$$P(k+1) : 1 + r + r^2 + r^3 + \ldots\ldots + r^k + r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1}$$

$$= \frac{1-r^{k+1}}{1-r} + \frac{r^{k+1}}{1}$$

$$= \frac{1-r^{k+1}(1)}{1-r(1)} + \frac{r^{k+1}(1-r)}{1(1-r)}$$

$$= \frac{1-r^{k+1}}{1-r} + \frac{r^{k+1} - r^{k+2}}{1-r}$$

$$= \frac{(1-r^{k+1}) + (r^{k+1} - r^{k+2})}{1-r}$$

$$= \frac{1-r^{k+1} + r^{k+1} - r^{k+2}}{1-r}$$

$$= \frac{1-r^{k+2}}{1-r}$$

$\therefore$ the proposition is true for $n = k + 1$ from 1, 2 and 3 :

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + r^k = \frac{(1-r^{k+2})}{1-r}$$

Proof by induction

$1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 1 \quad n \geq 0$

Solution:

1. for n = 0

$P(0) = 2^{n+1} - 1$

$P(0) = 2^{0+1} - 1$

$P(0) = 2 - 1$

$P(0) = 1$

$\therefore$ the proposition is true for n = 0

$P(k) = 2^{k+1} - 1$

$P(k+1) = 2^{k+1+1} - 1$

$= 2^{k+2} - 1$

2. suppose the proposition is true for n = k

$1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^k = 2^{k+1} - 1$

$P(k) = 2^{k+1} - 1$

And

$1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$P(k) = (2^{k+1} - 1) + 2^{k+1}$

$= 2(2^{k+1}) - 1$

$= 2^1 2^{k+1} - 1$

$= 2^{k+2} - 1$

$\therefore$ the proposition is true for $n \geq 0$

Example:

Proof by induction

$(A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cap \ldots\ldots\ldots \cap A_n^c$

For any collection of sets $A_1, A_2, A_3, \ldots\ldots\ldots\ldots\ldots\ldots, A_n$

Proof:

1. P (1) : " $A_1^c = A_1^c$ " is true.
2. Suppose P (k) is true, that is

$$\underbrace{(A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_k)^c}_{} = \underbrace{A_1^c \cap A_2^c \cap A_3^c \cap \ldots\ldots\ldots \cap A_k^c}_{\text{Suppose B}}$$

3.

$$P(k+1) : \text{"}(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{k+1})^c = \underbrace{A_1^c \cap A_2^c \cap A_3^c \cap \ldots \cap A_k^c}_{\text{This B}} \cap A_{k+1}^c\text{"}$$

$= (A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_k)^c \cap A_{k+1}^c$

$= A_1^c \cap A_2^c \cap A_3^c \cap \ldots \cap A_k^c \cap A_{k+1}^c$

$= A_1^c \cap A_2^c \cap A_3^c \cap \ldots\ldots\ldots\ldots \cap A_{k+1}^c$

From 1, 2, 3 we have $(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cap \ldots \cap A_n^c$

Example:

Let Q (n) be the predicate
2 + 4 + 6 + ………………………. + 2n = n (n+ 1)
Prove by induction that Q (n) is true for all n > 0.

Proof:

1. for n = 1

Q (1) = is the proposition
2 = n (n +1) ←→ 2 = 1 (1 +1) ←→ 2 = 2 which is true.
Q (2) = is the proposition
2+4 = n (n +1) ←→ 2 = 2 (2 +1) ←→ 6 = 6 which is true.
Q (k) = is the proposition
k = n (n +1) ←→ $\sum_{x=1}^{k} 2x$ = k (k +1) which is true.
Q (k+1) = is the proposition
K+1 = n (n +1) ←→ $\sum_{x=1}^{k+1} 2x$ = k+1 (k +2) which is true.

2. suppose that Q (k) is true Q (k) is the proposition

2 + 4 + 6 + ………………………. + 2k = k (k+ 1)

3. Q (k+1) is the proposition

2 + 4 + 6 + ………………………. + 2(k+1) = (k+ 1) (k +2)
2 + 4 + 6 + ……… 2(k+1) = $\underbrace{2 + 4 + 6 + \ldots + 2k}_{= k(k+1)}$ + 2(k+1)

= k (k+ 1) + 2(k+1)
= **(k+ 1)** (k +2)

∴ from 1, 2, 3 Q (n) is true for all n >0.